Math 122 Wednesday, October 26

G group, S a set. We say G *acts on* S if there is a map $G\times S\to S$ $(g,s)\longmapsto g(s)$ where each $g: S\to S$ is a bijective map. Also we require that i) $e(s)=s$ $\forall s\in S$ and ii) $gh(s)=g(h(s))$.

Equivalently, there is a group action of G on S $\Longleftrightarrow$ there is a homomorphism $f: G\to \mathrm{Aut}(S)$ of groups. Condition i) gives $f(e)=e'$ and ii) gives the homomorphism.

The *orbit* $O_s^{\subset S}$ of any element $s\in S$ is the set $\{s'\in S \mid g(s)=s' \text{ for some } g\in G\}$. There is an equivalence relation on S given by $s\equiv s'$ iff $\exists g\in G: g(s)=s'$ (note also $g^{-1}(s')=s$). Then O_s is the equivalence class of s and $S=\bigcup_{\text{orbits}} O_s$. If $S=O_s$ for a *single* orbit we say that G acts *transitively* on S.

ex. $S=G$, $g(s)=g\cdot s$ left multiplication. Then $O_e=S \Rightarrow$ action is transitive

The *stabalizer* $G_s < G$ of $s\in S$ is the set $\{h\in G \mid h(s)=s\}$. So for above ex. $G_g=G_e=\{e\}$

Claim $G/G_s \longleftrightarrow O_s$ as sets with an action of G.

$gG_s \longleftrightarrow s'=g(s)$

If G acts *transitively* on S with $G_s=\{e\}$ we say the action is *simply-transitive*. This gives an identification of G and S as sets by $g\mapsto g(s)$ for fixed s.

ex. $G=M(2)$ acts on the set $S=\{\text{all triangles } s=ABC \text{ in } \mathbb{R}^2\}$

$O_{ABC}=\{\text{all triangles } A'B'C' \text{ congruent to } ABC\}$

$G_{ABC}=\begin{cases} e & \text{if not isosceles} \\ S_2 & \text{if isosceles, not equilateral} \\ S_3 & \text{if equilateral}\end{cases}$

ex $M(2)$ also acts on set $\mathbb{R}^2=$ pts in $\mathbb{R}^2$. $O_0=\mathbb{R}^2 \Rightarrow$ transitive. $G_0=O(2)$

ex G acts on $G=S$ by conjugation $g(s)=gsg^{-1}$ $O_s=$ Conjugacy class of s.

$G_s=\{g\in G \mid gsg^{-1}=s\}=\{g\in G \mid gs=sg\}=$ the *centralizer* of s in $G > (Z_G \cup \langle s\rangle)$

for S_3 the conjugacy classes are $\{e\}$, $\{(12),(23),(13)\}$, $\{(123),(132)\}$

ex $G=S_n$ certainly acts on $\{1,2,3,\dots,n\}$ transitively. Note $G_k=S_{n-1}$. Also acts on pairs of elements $(a,b)\in S\times S$ but *not* transitively $O_{(1,1)}=\{(a,a)\}$, $O_{(1,2)}=\{(a,b)\mid a\neq b\}$.

Prop (from last time) If $\Gamma\subset M(n)$ is finite then there is a point $p\in\mathbb{R}^n$ fixed by all elements $g\in\Gamma$!

Cor Γ is conjugate to a finite subgroup of $O(n)=M(n)_0$

Pf: In fact, the point $p = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} g(s)$ for any $s \in \mathbb{R}^n$ is fixed. This can be called the center of gravity of the finite set of points $\{g(s)\}$. For $\{s_1, \dots, s_k\} \in \mathbb{R}^n$ define $p = p\{s_1, \dots, s_k\} = \frac{1}{k} \sum_{i=1}^{k} s_i \in \mathbb{R}^n$. Let $g \in M(n)$ $\{s_1' = g(s_1), s_2' = g(s_2), \dots, s_k' = g(s_k)\} \rightsquigarrow p' = g(p)$. Must prove this for $g =$ translation by $b \in \mathbb{R}^n$ and $g =$ orthogonal translation $A \in O(n)$. Say $s_i' = s_i + b$. Then $p' = \frac{1}{k} \sum (s_i + b) = \frac{1}{k} \sum s_i + \frac{1}{k} \cdot k \cdot b = p + b = g(p)$. Say $s_i' = A s_i$. Then $p' = \frac{1}{k} \sum A s_i = \frac{1}{k} A(\sum s_i) = A(\frac{1}{k} \sum s_i) = Ap = g(p)$. Now since each $\gamma \in \Gamma$ permutes the $\{g(s)\}$ we find that $\gamma p = p$ for all $\gamma \in \Gamma$.

Last time Finite subgroups $\Gamma \subset O(2)$ either cyclic (if $\Gamma \subset SO(2)$) or dihedral. These are the finite subgroups of $M(2)$ $\leftarrow$ no translations in a finite subgroup.

defn $\Gamma < M(2)$ is discrete if it doesn't contain an arbitrarily small rotation $r(\theta)$ or translation t_b.

ex. $\Gamma = \{t_n : n \in \mathbb{Z}\} \simeq (\mathbb{Z}, +)$. $t_m(v) = v + m$ $t_n \circ t_m = t_{n+m}$

If Γ is discrete the lattice of translations in Γ $L = \Gamma \cap \mathbb{R}^n \triangleleft \Gamma$ is in fact a lattice (doesn't contain arbitrarily small elements). "translations in $M(2)$"
$L = \mathbb{Z}$ is the above example.

Let $\bar{S}$ be a bounded set in $\mathbb{R}^2$. Then $L \cap \bar{S}$ is a finite set. If not, there is a sequence $v_1, v_2, \dots$ of elements of $L \cap \bar{S}$. By boundedness, there is a convergent subsequence so $\exists\, t_b, t_{b'} \in L$ $|b - b'| < \varepsilon$ for any $\varepsilon \Rightarrow t_b \circ t_{b'}^{-1} = t_{b-b'} \in L$ arbitrarily small.

Prop If $L \subset \mathbb{R}^2$ is a lattice ($\Gamma \cap \mathbb{R}^2$) then there are three possibilities:
(1) $L = \{0\}$ (Γ is finite)
(2) $L = a\mathbb{Z}$ $a \neq 0$
(3) $L = a\mathbb{Z} + b\mathbb{Z}$ where $\{a, b\}$ give a basis for $\mathbb{R}^2$.

Pf: (1) is possible. Assume L contains a non-zero vector v. Either $L \subset \mathbb{R}\cdot v$ or $L \not\subset \mathbb{R}\cdot v$. Assume $L \subset \mathbb{R}\cdot v$. Choose one of the two vectors on the line $\mathbb{R}v$ closest to the origin. Call this a. Clearly $\mathbb{Z}a \subset L$ as L is a group and $a \in L$. For any $w \in L$ say $w \neq ka$ for some $k \in \mathbb{Z}$. Choose $k \in \mathbb{Z}$ such that $ka \lneq w \lneq (k+1)a$. Then $|w - ka| < |(k+1)a - ka| = |a|$ contradicting the minimality of a. So $w = ka$ and $L \subset a\mathbb{Z} \Rightarrow L = a\mathbb{Z}$.
Finish with case $L \not\subset \mathbb{R}v$ next time.